POLITÉCNICA "Engineering the future" | INTERNATIONAL CAMPUS OF EXCELLENCE

08 – Flutter Phenomenon

PART 2 – SECTION 2:

- **CONTINUATION WITH THE FORMULATION OF THE UNSTEADY AERODYNAMIC FORCES**

  + GENERAL FORMULATION

  + 2D "TYPICAL SECTION" IN INCOMPRESSIBLE FLOW

- **REVIEW OF FLUTTER EQUATION WITH AERO FORCES IN THE FREQUENCY DOMAIN**

Vibraciones y Aeroelasticidad

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ESCUELA TÉCNICA SUPERIOR DE INGENIERÍA AERONÁUTICA Y DEL ESPACIO

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"When a flow is both frictionless and irrotational, pleasant things happen." –F.M. White, Fluid Mechanics 4th ed.

Once viscous effects are neglected …

Equation of continuity: $\dfrac{D\rho}{Dt} + \rho\nabla.\mathbf{q} = 0$

Momentum equation: $\dfrac{D\mathbf{q}}{Dt} + \dfrac{1}{\rho}\nabla p = 0$

Energy equation: $\dfrac{D}{Dt}\left(\dfrac{a^2}{\gamma - 1} + \dfrac{q^2}{2}\right) = \dfrac{1}{\rho}\dfrac{\partial p}{\partial t}$

Equation of state: $p = \rho RT$

Circulation: $\Gamma = \oint q \cdot ds.$

The Kelvin's theorem: $\dfrac{D\Gamma}{Dt} = -\oint \dfrac{dp}{\rho}.$

For incompressible flow or a barotropic flow where $p = p(\rho)$ the right hand side of Kelvin's theorem vanishes to yield

$$\frac{D\Gamma}{Dt} = 0.$$

This tells us that the circulation under these conditions remains the same with time. Now, let us analyze the flow with constant free stream which is the most referred flow case in aerodynamics. Since the free stream is constant then its circulation $\Gamma = 0$. The Stokes theorem states that

$$\oint q \cdot ds = \oiint \nabla \times \mathbf{q} \cdot \mathbf{dA} = 0 \tag{2.5}$$

The integrand of the double integral must be zero in order to have Eq. 2.5 equal to zero for arbitrary differential area element. This gives $\nabla \times \mathbf{q} = 0$.

$\nabla \times \mathbf{q} = 0$, on the other hand, implies that the velocity vector $\mathbf{q}$ can be obtained from the gradient of a scalar potential ϕ, i.e.

$$\mathbf{q} = \nabla\,\phi \tag{2.6}$$

Ref.: Chapter 2 of "Fundamentals of Unsteady Aerodynamics" Ülgen Gülçat

COMPRESSIBLE FLOW DIFFERENTIAL EQUATION

Mass-conservation equation:

$$\frac{\partial \rho}{\partial t} + \vec{V}\cdot\nabla\rho + \rho\nabla\cdot\vec{V} = 0 \Rightarrow \underbrace{\boxed{\frac{1}{\rho}\frac{\partial \rho}{\partial t}}}_{1} + \underbrace{\boxed{\nabla\Phi\frac{\nabla\rho}{\rho}}}_{2} + \nabla^2\Phi = 0$$

Assumption of Irrotational Flow

$$\nabla\times\vec{V} = \nabla\times\nabla\Phi = 0$$

Equations of motion w/o viscosity:

$$\vec{V}\cdot\nabla u = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} = \frac{\partial\left(\vec{V}\cdot\vec{V}\right)}{\partial x}$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \vec{V}\cdot\nabla u = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + \vec{V}\cdot\nabla v = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + \vec{V}\cdot\nabla w = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\frac{\partial\vec{V}}{\partial t} + \nabla\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) + \frac{\nabla p}{\rho} \Rightarrow \nabla\left(\frac{\partial\Phi}{\partial t} + \frac{\vec{V}\cdot\vec{V}}{2} + \int\frac{dp}{\rho}\right) = 0 \Rightarrow \frac{\partial\Phi}{\partial t} + \frac{\vec{V}\cdot\vec{V}}{2} + \int\frac{dp}{\rho} = F(t)$$

1) $$\frac{\partial}{\partial t}: \quad \frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial t} + \frac{\vec{V}\cdot\vec{V}}{2} + \int\frac{dp}{\rho}\right) = 0 \Rightarrow -\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial}{\partial t}\left(\frac{V^2}{2}\right) = \frac{1}{\rho}\frac{\partial p}{\partial t} = \frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial\rho}{\partial t} = \frac{a^2}{\rho}\frac{\partial\rho}{\partial t}$$

2) $$\nabla: \quad \nabla\left(\frac{\partial\Phi}{\partial t} + \frac{\vec{V}\cdot\vec{V}}{2} + \int\frac{dp}{\rho}\right) \Rightarrow -\nabla\left(\frac{\partial\Phi}{\partial t} + \frac{\vec{V}\cdot\vec{V}}{2}\right) = \frac{\nabla p}{\rho} = \frac{dp}{d\rho}\frac{\nabla\rho}{\rho} = a^2\frac{\nabla\rho}{\rho}$$

$$\vec{V}\cdot\frac{\nabla\rho}{\rho} = -\frac{1}{a^2}\left[\vec{V}\cdot\nabla\left(\frac{\partial\Phi}{\partial t}\right) + \vec{V}\cdot\nabla\left(\frac{V^2}{2}\right)\right] = -\frac{1}{a^2}\left[\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right) + \vec{V}\cdot\nabla\left(\frac{V^2}{2}\right)\right]$$

$$-\frac{1}{a^2}\left[\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\frac{V^2}{2}\right)\right] - \frac{1}{a^2}\left[\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right) + \vec{V}\cdot\nabla\left(\frac{V^2}{2}\right)\right] + \nabla^2\Phi = 0$$

$$\boxed{\nabla^2\Phi - \frac{1}{a^2}\left[\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial V^2}{\partial t} + \vec{V}\cdot\nabla\left(\frac{V^2}{2}\right)\right] = 0}$$

COMPRESSIBLE FLOW DIFFERENTIAL EQUATION →

$$\nabla^2\Phi - \frac{1}{a^2}\left[\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial V^2}{\partial t} + \vec{V}\cdot\nabla\left(\frac{V^2}{2}\right)\right] = 0$$

LINEARIZATION (STEADY+UNSTEADY) →

$$\Phi(x,z;t) = U_\infty x + \epsilon_0\phi_0(x,z) + \epsilon_1\phi_1(x,z;t)$$

$$\nabla^2\Phi = (\phi_{0xx} + \phi_{0zz})\,\epsilon_0 + (\phi_{1xx} + \phi_{1zz})\,\epsilon_1$$

$$a^2 \approx a_\infty^2$$

$$\frac{\partial^2\Phi}{\partial t^2} = \phi_{1tt}\epsilon_1$$

$$\frac{\partial V^2}{\partial t} = 2U_\infty\phi_{1xt}\epsilon_1$$

$$\vec{V}\cdot\nabla\left(\frac{V^2}{2}\right) \approx U_\infty^2\phi_{0xx}\epsilon_0 + U_\infty^2\phi_{1xx}\epsilon_1$$

$$(\phi_{0xx} + \phi_{0zz})\,\epsilon_0 + (\phi_{1xx} + \phi_{1zz})\,\epsilon_1 - \frac{1}{a_\infty^2}\left(\phi_{1tt}\epsilon_1 + 2U_\infty\phi_{1xt}\epsilon_1 + U_\infty^2\phi_{0xx}\epsilon_0 + U_\infty^2\phi_{1xx}\epsilon_1\right) = 0$$

Small perturbation method:

Thickness & Steady problem (ε_0)

+

Unsteady problem (ε_1)

$$\Downarrow$$

STEADY PROBLEM → $\left(1 - M_\infty^2\right)\phi_{0xx} + \phi_{0zz} = 0$

UNSTEADY PROBLEM →

$$\left(1 - M_\infty^2\right)\phi_{1xx} + \phi_{1zz} - \frac{2U_\infty}{a_\infty^2}\phi_{1xt} - \frac{1}{a_\infty^2}\phi_{1tt} = 0 \xrightarrow{a_\infty \to \infty} \boxed{\nabla^2\phi_1 = 0}$$

Equation of the airfoil surface:

$$F(x,z;t) = z - z_0(x)\epsilon_0 - z_1(x;t)\epsilon_1 = 0$$

Condition of flow in permanent contact with the surface:

$$\frac{DF}{Dt} = -z_{1t}\epsilon_1 + \nabla\Phi \cdot \nabla F = -z_{1t}\epsilon_1 + (U_\infty + \phi_{0x}\epsilon_0 + \phi_{1x}\epsilon_1, \phi_{0z}\epsilon_0 + \phi_{1z}\epsilon_1) \cdot (-z_{0x}\epsilon_0 - z_{1x}\epsilon_1, 1) = 0$$

$$\phi_{0z}(x,0) \approx U_\infty z_{0x}(x)$$

$$\boxed{\phi_{1z}(x,0;t) \approx U_\infty z_{1x}(x;t) + z_{1t}(x;t)}$$ Unsteady-flow condition

$$\frac{\partial \Phi}{\partial t} = \epsilon_1 \phi_{1t}$$

$$\frac{\partial \Phi}{\partial t} + \frac{(\nabla \Phi)^2}{2} + \int \frac{dp}{\rho} = cte.$$

$$\nabla \Phi = (U_\infty + \epsilon_0 \phi_{0x} + \epsilon_1 \phi_{1x}, \epsilon_0 \phi_{0z} + \epsilon_1 \phi_{1z}) \qquad (\nabla \Phi)^2 = U_\infty^2 + 2U_\infty \phi_{0x}\epsilon_0 + 2U_\infty \phi_{1x}\epsilon_1$$

$$\int \frac{dp}{\rho} = \frac{p_\infty}{\rho_\infty} \int \frac{d\left(\frac{p}{p_\infty}\right)}{\frac{\rho}{\rho_\infty}} = \frac{p_\infty}{\rho_\infty} \int \frac{d\left(\frac{p}{p_\infty}\right)}{\left(\frac{p}{p_\infty}\right)^{1/\gamma}} = \frac{p_\infty}{\rho_\infty} \frac{\gamma}{\gamma - 1} \left(\frac{p}{p_\infty}\right)^{-1/\gamma + 1} = \frac{a_\infty^2}{\gamma - 1} \left(\frac{p}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\epsilon_1 \phi_{1t} + \frac{U_\infty^2}{2} + U_\infty \phi_{0x}\epsilon_0 + U_\infty \phi_{1x}\epsilon_1 + \frac{a_\infty^2}{\gamma - 1}\left(\frac{p}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}} = \frac{U_\infty^2}{2} + \frac{a_\infty^2}{\gamma - 1}$$

$$\frac{a_\infty^2}{\gamma - 1}\left[\left(\frac{p}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] = -U_\infty \phi_{0x}\epsilon_0 - (\phi_{1t} + U_\infty \phi_{1x})\,\epsilon_1$$

$$\left(\frac{p}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}} = 1 + \frac{\gamma - 1}{a_\infty^2}\left[-U_\infty \phi_{0x}\epsilon_0 - (\phi_{1t} + U_\infty \phi_{1x})\,\epsilon_1\right]$$

Expression used when non-linear effects need to be captured

$$\frac{p}{p_\infty} = \left\{1 + \frac{\gamma - 1}{a_\infty^2}\left[-U_\infty \phi_{0x}\epsilon_0 - (\phi_{1t} + U_\infty \phi_{1x})\,\epsilon_1\right]\right\}^{\frac{\gamma}{\gamma-1}} \approx 1 - \frac{\gamma}{a_\infty^2}\left[U_\infty \phi_{0x}\epsilon_0 + (\phi_{1t} + U_\infty \phi_{1x})\,\epsilon_1\right]$$

$$\frac{p - p_\infty}{p_\infty} = -\frac{\gamma}{a_\infty^2}\left[U_\infty \phi_{0x}\epsilon_0 + (\phi_{1t} + U_\infty \phi_{1x})\,\epsilon_1\right]$$

$$\frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = C_{p0}\epsilon_0 + C_{p1}\epsilon_1 = -\frac{2}{U_\infty^2}\left[U_\infty \phi_{0x}\epsilon_0 + (\phi_{1t} + U_\infty \phi_{1x})\,\epsilon_1\right]$$

$$C_{p0} = -\frac{2}{U_\infty}\phi_{0x}$$

$$\boxed{C_{p1} = -\frac{2}{U_\infty^2}(\phi_{1t} + U_\infty \phi_{1x})}$$

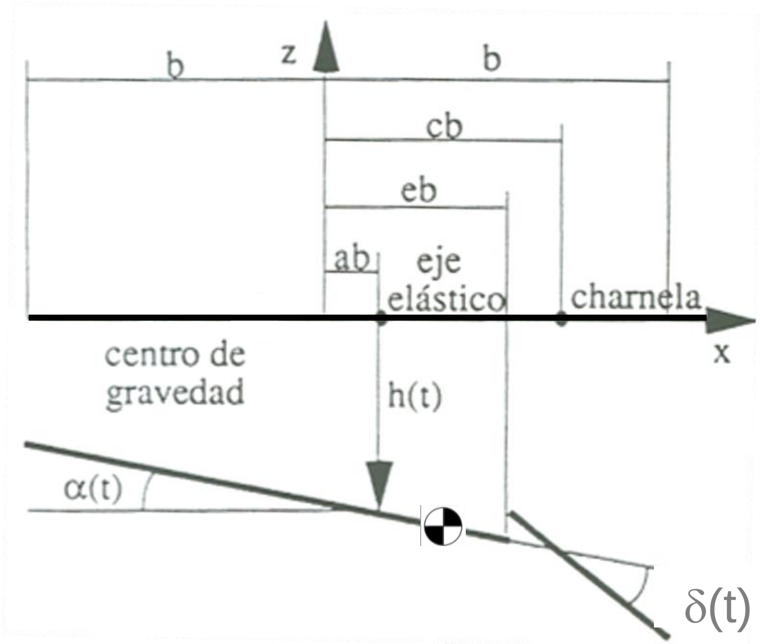
1935: Theodore Theodorsen

REPORT No. 496

GENERAL THEORY OF AERODYNAMIC INSTABILITY AND THE MECHANISM OF FLUTTER

By Theodore Theodorsen

- In Theodorsen's approach, only three major simplifications are assumed:
  - The flow is always attached, i.e. the motion's amplitude is small
  - The wing is a flat plate
  - The wake is flat
- The flat plate assumption is not problematic. In fact Theodorsen worked on a flat plate with a control surface (3 d.o.f.s), so asymmetric wings can also be handled.
- If the motion is small (first assumption) then the flat wake assumption has little influence on the results.



UNSTEADY INCOMPRESSIBLE FLOW

DIFFERENTIAL EQUATION + BOUNDARY CONDITIONS

$$\boxed{\nabla^2 \phi_1 = 0}$$

- Boundary condition: $F(x, z; t) = z - \epsilon_0 z_0(x) - \epsilon_1 z_1(x; t)$

  (*) WING IS A FLAT PLATE

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \nabla \Phi \cdot \nabla F \Rightarrow \boxed{\phi_{1z}(x, 0) = z_{1t} + U_\infty z_{1x}}$$

- Far Field: $\boxed{\phi_1 \to 0 \qquad x^2 + z^2 \to \infty}$

- Kutta condition at trailing edge: (*) ATTACHED FLOW HYPOTHESIS VALID FOR SMALL MOTIONS

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla \Phi)^2 + \frac{p}{\rho_\infty} = cte$$

$$C_p = \epsilon_0 \boxed{C_{p0}} + \epsilon_1 \boxed{C_{p1}}$$

$$C_{p0} = -2\frac{\phi_{0x}}{U_\infty}$$

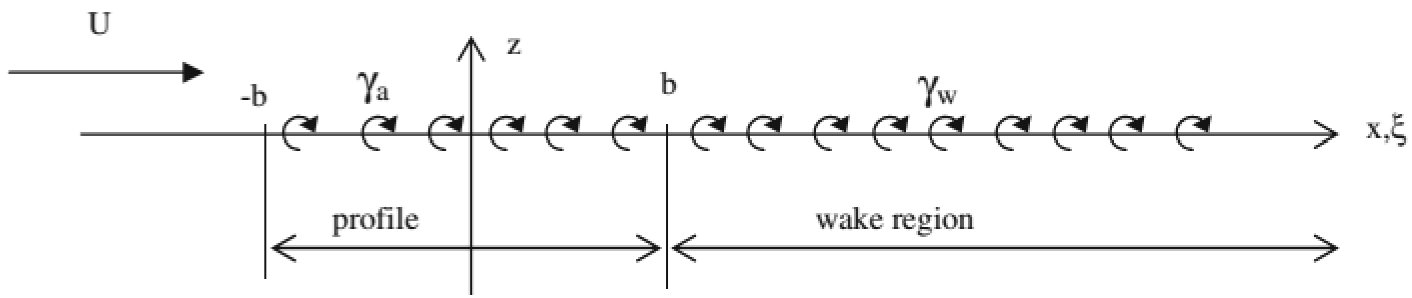
$$C_{p1} = -\frac{2}{U_\infty^2}\left(\phi_{1t} + U_\infty \phi_{1x}\right)$$

$$\Delta C_{p1}(b, 0) = 0 \Rightarrow \boxed{\Delta \phi_{1t}(b, 0) + U_\infty \Delta \phi_{1x}(b, 0) = 0}$$

A 2D Vortex singularity "$\gamma \cdot dx$" is an elementary solution of $\nabla^2\phi_1=0$ as the potential "$d\phi_1$" written below that is induced by the 2D vortex satisfies the Laplace equation:

$$d\phi_1 = -\frac{\gamma(\xi)\,dx}{2\pi}\theta = -\frac{\gamma dx}{2\pi}\arctan\left(\frac{z}{x-\xi}\right)$$

Vortices are distributed along the 2D section and the wake:



OVERVIEW OF THE THEODORSEN'S APPROACH

$$\gamma_a(\xi,t) = \bar{\gamma}_a(\xi)e^{i\omega t}$$

$$\gamma_w(\xi,t) = \bar{\gamma}_w(\xi)e^{i\omega t}$$

$$\varphi_1(x,z,t) = \bar{\varphi}_1(x,z)e^{i\omega t}$$

$$\varphi_{1z}(x,0,t) = -\frac{1}{2\pi}\int_{-b}^{b}\frac{\gamma_a(\xi,t)}{x-\xi}d\xi - \boxed{\frac{1}{2\pi}\int_{b}^{\infty}\frac{\gamma_w(\xi,t)}{x-\xi}d\xi} \longrightarrow \bar{\varphi}_{1z}(x,0) = -\frac{1}{2\pi}\int_{-b}^{b}\frac{\bar{\gamma}_a(\xi)}{x-\xi}d\xi - \frac{1}{2\pi}\int_{b}^{\infty}\frac{\bar{\gamma}_w(\xi)}{x-\xi}d\xi$$

(*) FLAT WAKE IS IMPLICITLY ASSUMED

$$\bar{\gamma}_w(\xi)e^{i\omega t} = -\frac{1}{U_\infty}i\omega\bar{\Gamma}e^{i\omega\left(t-\frac{(\xi-b)}{U_\infty}\right)}$$

$$\bar{w}(x,0;k) = ik\bar{z}_1(x) + \frac{d\bar{z}_1(x)}{dx} = -\frac{1}{2\pi}\int_{-1}^{1}\frac{\bar{\gamma}_a(\xi;k)}{x-\xi}d\xi + \frac{ik\bar{\Gamma}e^{ik}}{2\pi}\int_{1}^{\infty}\frac{e^{-ik\xi}}{x-\xi}d\xi$$

Goldstein

$$\bar{\gamma}_a(x;k) = \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\left[\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{\bar{w}(\xi,0;k)}{x-\xi}d\xi + \frac{ik\bar{\Gamma}e^{ik}}{2}\int_{1}^{\infty}\frac{e^{-ik\lambda}}{x-\lambda}\sqrt{\frac{\lambda+1}{\lambda-1}}d\lambda\right]$$

$$\int_{-1}^{1}\gamma_a(x)dx = \bar{\Gamma} = -\frac{2}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{w(\xi,0,k)}{\xi-x}d\xi dx - \frac{ik\bar{\Gamma}}{\pi}e^{ik}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\int_{1}^{\infty}\frac{e^{-ik\lambda}}{\lambda-x}\sqrt{\frac{\lambda+1}{\lambda-1}}d\lambda dx$$

$$\longrightarrow \bar{\Gamma} = \frac{4\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}w(\xi,0,k)d\xi}{\pi e^{ik}ik\left[H_1^{(2)}(k)+iH_0^{(2)}(k)\right]} \Rightarrow \boxed{\bar{\gamma}_a(x;k)}$$

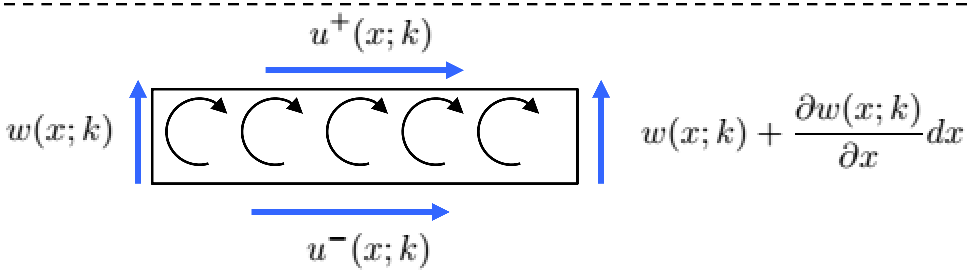
ΔC_{p1} AS FUNCTION OF THE RUNNING CIRCULATION γ_a

$$\frac{\partial \Phi}{\partial t}+\frac{1}{2}\left|\nabla \Phi\right|^2+\frac{p}{\rho_\infty}=\frac{1}{2}U_\infty^2+\frac{p_\infty}{\rho_\infty}\Rightarrow C_{p1}=-\frac{2}{U_\infty^2}\left(\phi_{1t}+U_\infty\phi_{1x}\right)$$

$$\Delta C_{p1}=C_{p1}^L-C_{p1}^U=\frac{2}{U_\infty^2}\left[U_\infty\left(\phi_{1x}^U-\phi_{1x}^L\right)+\phi_{1t}^U-\phi_{1t}^L\right]$$

$$\Delta\bar{C}_{p1}=\frac{2}{U_\infty^2}\left[U_\infty\left(\bar{\phi}_{1x}^U-\bar{\phi}_{1x}^L\right)+i\omega\left(\bar{\phi}_{1}^U-\bar{\phi}_{1}^L\right)\right]=\boxed{\frac{2}{U_\infty^2}\left(U_\infty\bar{\gamma}_a(x;k)+i\omega\int_{-b}^{x}\bar{\gamma}_a(x;k)dx\right)}$$

$$\bar{\phi}_1^U(x;k)-\bar{\phi}_1^L(x;k)=\int_{-\infty}^{x}[\phi_{1x}^U(\xi;k)-\phi_{1x}^L(\xi;k)]d\xi=\int_{-b}^{x}\gamma_a(\xi;k)d\xi$$



$$\gamma_a(x;t)dx=u(x;t)^+dx-\left(w(x;t)+\frac{\partial w(x;t)}{\partial x}dx\right)dz-u^-(x;t)dx+w(x;t)dz\approx=$$

$$\approx\left(u^+(x;t)-u^-(x;t)\right)dx\Rightarrow\Delta\phi_{1x}(x;t)=\gamma_a(x;t)\Rightarrow\Delta\bar{\phi}_{1x}(x;k)=\bar{\gamma}_a(x;k)$$

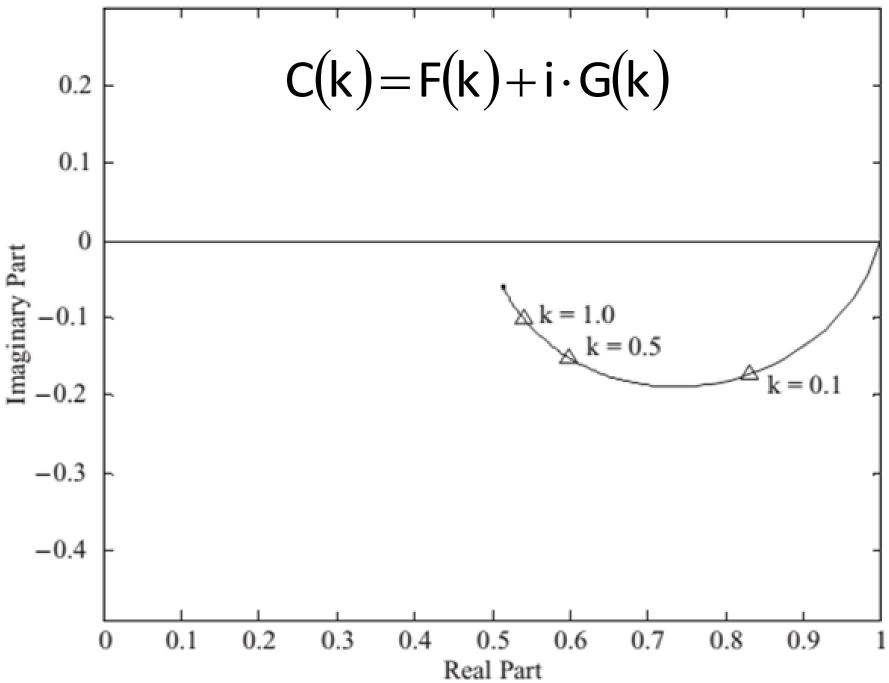
ΔC_{p1} AS FUNCTION OF THE AIRFOIL MOTION

$$\Delta\overline{C}_{p1} = 2\left[\overline{\gamma}_a(x;k) + ik\int_{-1}^{x}\overline{\gamma}_a(\xi;k)\,d\xi\right] \rightarrow$$

$$\Delta C_{P1}(x,k) = \frac{4}{\pi}\int_{-1}^{1} w(\xi;k)\left\{\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+\xi}{1-\xi}}\frac{1}{x-\xi} - ik\Lambda_1(x,\xi)\right\}d\xi$$

$$+\frac{4}{\pi}(1-C(k))\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,w(\xi,k)\,d\xi$$

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)}$$



$$\Lambda_1(x,\xi) = \frac{1}{2}\ln\left[\frac{1-x\xi+\sqrt{1-\xi^2}\sqrt{1-x^2}}{1-x\xi-\sqrt{1-\xi^2}\sqrt{1-x^2}}\right]$$

$\Delta\overline{C}_{p1}$

for the boundary condition

$$\overline{w}(x,0;k) = ik\overline{z}_1(x) + \frac{d\overline{z}_1(x)}{dx}$$

Note: $H_n^{(2)}(k)$ are Hankel functions of the second kind that are expressed in terms of Bessel functions of the first and second type as $H_n^{(2)}(k) = J_n(k) - i\cdot Y_n(k)$

GENERALIZED FORCES AS INTEGRAL OF ΔC_{p1}

- Boundary condition (vertical speed): $\bar{w}(\xi;k) = ik \cdot \bar{z}_1(\xi) + U_\infty \dfrac{d\bar{z}_1(\xi)}{d\xi}$

- Vertical displacement (to obtain $\bar{w}(\xi;k)$):

$$\bar{z}_1(\xi;k) = -\bar{h}(k) - \bar{\alpha}(k)\,(\xi - a) - \bar{\delta}(k)\,(\xi - c)\,H\,(\xi - e)$$

- Substituting in the equation for $\Delta\bar{C}_{p1}$ of previous slide:

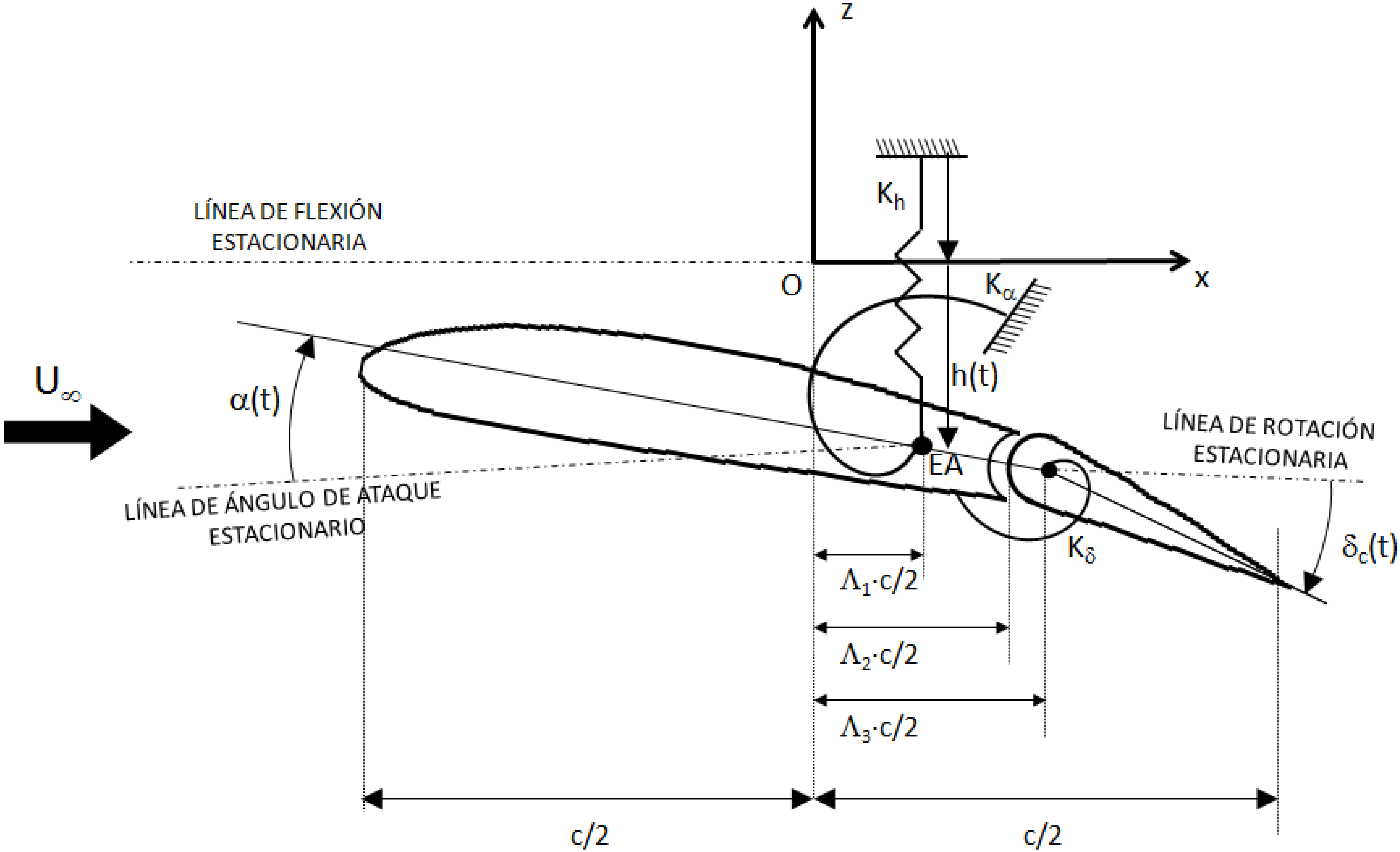
$$\Delta\bar{C}_{p1} = \Delta\bar{C}_{p1h} + \Delta\bar{C}_{p1\alpha} + \Delta\bar{C}_{p1\delta}$$

- The integration of $\Delta\bar{C}_{p1}$ leads to the generalized forces:

$$\bar{Q}_h = q_\infty b \int_{-1}^{1} \Delta\bar{C}_{p1}(-1)dx = q_\infty b\,(-C_L)$$

$$\bar{Q}_\alpha = q_\infty b^2 \int_{-1}^{1} \Delta\bar{C}_{p1}(a-x)dx = q_\infty b^2 C_{M\alpha}$$

$$\bar{Q}_\delta = q_\infty b^2 \int_{e}^{1} \Delta\bar{C}_{p1}(c-x)dx = q_\infty b^2 C_{M\delta}$$


z
x
O
K_h
K_α
$h(t)$
LÍNEA DE FLEXIÓN
ESTACIONARIA
U_∞
$\alpha(t)$
LÍNEA DE ÁNGULO DE ATAQUE
ESTACIONARIO
EA
LÍNEA DE ROTACIÓN
ESTACIONARIA
K_δ
$\delta_c(t)$
$\Lambda_1 \cdot c/2$
$\Lambda_2 \cdot c/2$
$\Lambda_3 \cdot c/2$
c/2
c/2

$$t_0 = c/2U_\infty$$
$$\Lambda_{31} = \Lambda_3 - \Lambda_1$$

NONCIRCULATORY TERMS = “APPARENT-MASS”

CIRCULATORY TERMS

$$\frac{\tilde{Q}_h(k)}{q_\infty \frac{c}{2}} = \boxed{-2\pi\left[\frac{t_0^2\ddot{h}}{c/2} + t_0\dot{\alpha} - \Lambda_1 t_0^2\ddot{\alpha} - \frac{T_4}{\pi}t_0\dot{\delta}_c - \frac{T_1}{\pi}t_0^2\ddot{\delta}_c\right]} -$$
$$\boxed{-4\pi\cdot C(k)\left\{\frac{t_0\dot{h}}{c/2} + \alpha + \left(\frac{1}{2}-\Lambda_1\right)t_0\dot{\alpha} + \frac{T_{10}}{\pi}\delta_c + \frac{T_{11}}{2\pi}t_0\dot{\delta}_c\right\}}$$

$$\frac{\tilde{Q}_\alpha(k)}{q_\infty\left(\frac{c}{2}\right)^2} = -2\pi\left\{-\Lambda_1\frac{t_0^2\ddot{h}}{c/2} + \left(\frac{1}{2}-\Lambda_1\right)t_0\dot{\alpha} + \left(\frac{1}{8}+\Lambda_1^2\right)t_0^2\ddot{\alpha} + \right.$$
$$\left. + \frac{T_4+T_{10}}{\pi}\delta_c + \frac{T_1 - T_8 - \Lambda_{31}T_4 + \frac{1}{2}T_{11}}{\pi}t_0\dot{\delta}_c - \frac{\Lambda_{31}T_1 + T_7}{\pi}t_0^2\ddot{\delta}_c\right\} + \tag{4.59}$$
$$+4\pi\left(\frac{1}{2}+\Lambda_1\right)C(k)\left\{\frac{t_0\dot{h}}{c/2} + \alpha + \left(\frac{1}{2}-\Lambda_1\right)t_0\dot{\alpha} + \frac{T_{10}}{\pi}\delta_c + \frac{T_{11}}{2\pi}t_0\dot{\delta}_c\right\}$$

$$\frac{\tilde{Q}_\delta(k)}{q_\infty\left(\frac{c}{2}\right)^2} = -\left\{-2T_1\frac{t_0^2\ddot{h}}{c/2} - 2\left(T_1 + T_4\left(\frac{1}{2}-\Lambda_1\right) + 2T_9\right)t_0\dot{\alpha} + 4T_{13}t_0^2\ddot{\alpha}\right.$$
$$\left. + \frac{2}{\pi}\left(T_5 - T_4T_{10}\right)\delta_c - \frac{T_4T_{11}}{\pi}t_0\dot{\delta}_c - 2\frac{T_3}{\pi}t_0^2\ddot{\delta}_c\right\} - \tag{4.60}$$
$$-2T_{12}C(k)\left\{\frac{t_0\dot{h}}{c/2} + \alpha + \left(\frac{1}{2}-\Lambda_1\right)t_0\dot{\alpha} + \frac{T_{10}}{\pi}\delta_c + \frac{T_{11}}{2\pi}t_0\dot{\delta}_c\right\}$$

T1 to T14: GEOMETRICAL CHARACTERISTICS OF SECTION

$$T_1 = -\frac{1}{3}\left(2+\Lambda_3^2\right)\sqrt{1-\Lambda_3^2} + \Lambda_3 \cdot \arccos\Lambda_3$$

$$T_2 = \Lambda_3\left(1-\Lambda_3^2\right) - \left(1+\Lambda_3^2\right)\sqrt{1-\Lambda_3^2}\arccos\Lambda_3 + \Lambda_3\left(\arccos\Lambda_3\right)^2$$

$$\begin{aligned} T_3 \;\; = \;\; & -\left(\frac{1}{8}+\Lambda_3^2\right)\left(\arccos\Lambda_3\right)^2 + \frac{1}{4}\Lambda_3\left(7+2\Lambda_3^2\right)\sqrt{1-\Lambda_3^2}\arccos\Lambda_3 \\ & -\frac{1}{8}\left(1-\Lambda_3^2\right)\left(5\Lambda_3^2+4\right) \end{aligned}$$

$$T_4 = -\arccos\Lambda_3 + \Lambda_3\sqrt{1-\Lambda_3^2}$$

$$T_5 = -\left(1-\Lambda_3^2\right) - \left(\arccos\Lambda_3\right)^2 + 2\Lambda_3\sqrt{1-\Lambda_3^2}\arccos\Lambda_3$$

$$T_6 = T_2$$

$$T_7 = -\left(\frac{1}{8}+\Lambda_3^2\right)\arccos\Lambda_3 + \frac{1}{8}\Lambda_3\left(7+2\Lambda_3^2\right)\sqrt{1-\Lambda_3^2}$$

$$T_8 = -\frac{1}{3}\left(2\Lambda_3^2+1\right)\sqrt{1-\Lambda_3^2} + \Lambda_3\arccos\Lambda_3$$

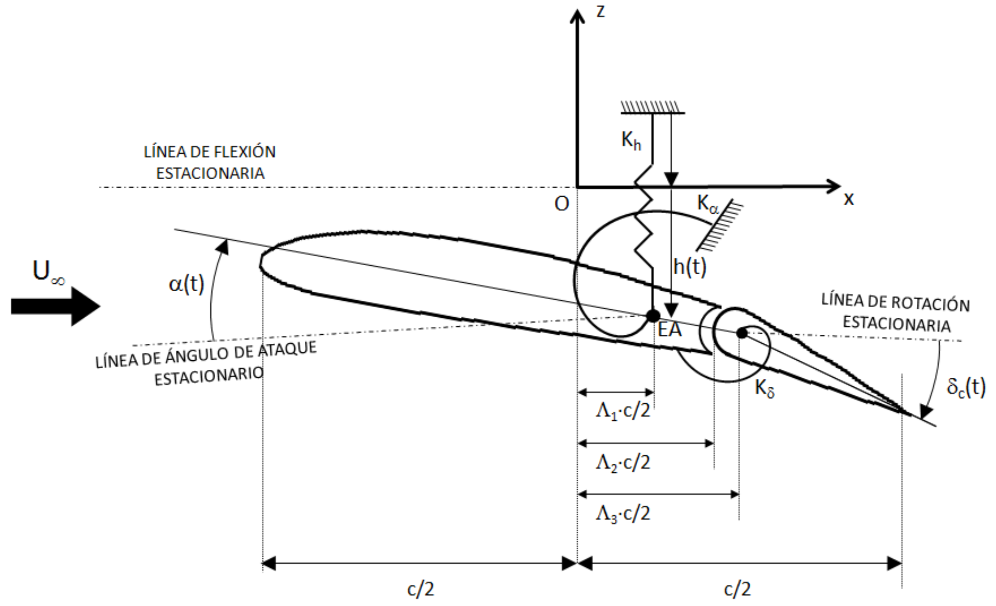
$$T_9 = \frac{1}{2}\left[\frac{1}{3}\left(1-\Lambda_3^2\right)^{3/2} + \Lambda_1 T_4\right]$$

$$T_{10} = \sqrt{1-\Lambda_3^2} + \arccos\Lambda_3$$

$$T_{11} = \left(1-2\Lambda_3\right)\arccos\Lambda_3 + \left(2-\Lambda_3\right)\sqrt{1-\Lambda_3^2}$$

$$T_{12} = \left(2+\Lambda_3\right)\sqrt{1-\Lambda_3^2} - \left(2\Lambda_3+1\right)\arccos\Lambda_3$$

$$T_{13} = -\frac{1}{2}\left(T_7 + \Lambda_{31} T_1\right) \qquad T_{14} = \frac{1}{16} + \frac{1}{2}\Lambda_1\Lambda_3$$



- Motion at low reduced frequency, i.e., k<<1.
- Fung (1983) approach:
  - Unsteadiness is retained exclusively in the boundary condition over the section.
  - Approach valid for incompressible or subsonic flow.

Ecuación diferencial	$\nabla^2\phi_1(x,z,t)=0$
Sobre el perfil $-c/2 \leq x \leq c/2$	$\phi_{1z}(x,0,t)=z_{1t}(x,t)+U_\infty z_{1x}(x,t)$
En el infinito $x^2+z^2\to\infty$	$\phi_1(x,z,t)\to 0$
En la estela $x \geq c/2$	$\Delta\phi_{1x}(x,0,t)=0$
Coeficiente de presión	$C_{p1}=-\frac{2}{U_\infty}\phi_{1x}(x,0,t)$

- The vortices distribution is written as:

$$\frac{\tilde{\gamma}_a}{U_\infty}(\hat{x};k)=\frac{2}{\pi}\sqrt{\frac{1-\hat{x}}{1+\hat{x}}}\left[\int_{-1}^{+1}\frac{\frac{\phi_{1z}}{U_\infty}\left(\hat{\xi},0;k\right)}{\hat{x}-\hat{\xi}}\sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}}d\hat{\xi}\boxed{+\frac{1}{2}ik\frac{\tilde{\Gamma}(k)}{\frac{c}{2}U_\infty}e^{ik}\int_{+1}^{+\infty}\frac{e^{-ik\hat{\xi}}}{\hat{x}-\hat{\xi}}\sqrt{\frac{\hat{\xi}+1}{\hat{\xi}-1}}d\hat{\xi}}\right]$$

- Vertical motion (plunging or heaving): $\frac{\tilde{z}_1}{c/2}(\hat{\xi};k) = -\frac{h_0}{c/2}$ $\quad \frac{\tilde{\phi}_{1z}}{U_\infty}(\hat{\xi},0;k) = -ik\frac{h_0}{c/2}$

$$\triangle\tilde{C}_{p1}(\hat{x};k) = -\frac{4}{\pi}\sqrt{\frac{1-\hat{x}}{1+\hat{x}}}\int_{-1}^{+1}\sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}}\frac{-ik\frac{h_0}{c/2}}{\hat{\xi}-\hat{x}}d\hat{\xi} = 4ik\frac{h_0}{c/2}\sqrt{\frac{1-\hat{x}}{1+\hat{x}}}$$

- Rotation/Torsion $\quad \frac{\tilde{z}_1}{c/2}(\hat{\xi};k) = \left(\Lambda_1 - \hat{\xi}\right)\tilde{\alpha}_0 \quad \frac{\tilde{\phi}_{1z}}{U_\infty}(\hat{\xi},0;k) = -\tilde{\alpha}_0 + ik\left(\Lambda_1 - \hat{\xi}\right)\tilde{\alpha}_0$

$$\begin{aligned}\triangle\tilde{C}_{p1}(\hat{x};k) &= -\frac{4}{\pi}\sqrt{\frac{1-\hat{x}}{1+\hat{x}}}\int_{-1}^{+1}\sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}}\frac{-\tilde{\alpha}_0 + ik\Lambda_1 - \hat{\xi})\tilde{\alpha}_0}{\hat{\xi}-\hat{x}}d\hat{\xi} = \\ &= -\frac{4}{\pi}\sqrt{\frac{1-\hat{x}}{1+\hat{x}}}\left[-1 + ik\Lambda_1 - ik(1+\hat{x})\right]\pi\tilde{\alpha}_0\end{aligned}$$

- Control surface rotation $\quad \frac{\tilde{z}_1}{c/2}(\hat{\xi};k) = \begin{cases} 0 & \hat{\xi} < \Lambda_2 \\ \left(\Lambda_3 - \hat{\xi}\right)\tilde{\delta}_c & \hat{\xi} > \Lambda_2 \end{cases} \quad \frac{\tilde{\phi}_{1z}}{U_\infty}(\hat{\xi},0;k) = \begin{cases} 0 & \hat{\xi} < \Lambda_2 \\ -\tilde{\delta}_c + ik\left(\Lambda_3 - \hat{\xi}\right)\tilde{\delta}_c & \hat{\xi} > \Lambda_2 \end{cases}$

$$\begin{aligned}\triangle\tilde{C}_{p1}(\hat{x};k) &= \frac{4}{\pi}\sqrt{\frac{1-\hat{x}}{1+\hat{x}}}\left\{\frac{\pi}{2} + \ln\frac{1+\sqrt{1-\hat{x}^2}}{\hat{x}}\right\}\tilde{\delta}_c + \qquad (4.80) \\ &+ \frac{4}{\pi}\left\{\frac{\pi}{2}\sqrt{\frac{1-\hat{x}}{1+\hat{x}}}(1+\hat{x}-\Lambda_3) + (\hat{x}-\Lambda_3)\ln\frac{1+\sqrt{1-\hat{x}^2}}{\hat{x}} + \sqrt{\frac{1-\hat{x}}{1+\hat{x}}}\right\}ik\hat{\delta}_c\end{aligned}$$

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